Maths Learning Service: Revision	*Mathematics IA*
Matrices	*Mathematics IMA*

A matrix is an array of numbers, written within a set of [ ] brackets, and arranged into a pattern of rows and columns. For example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 21 & 7 & -4 & 9 \end{bmatrix}$$

The **order** (or **size**, or **dimension**) of a matrix is written as "$m \times n$" where m = the number of rows, and n = the number of columns. For example, the matrices above have dimensions

$$A = 2 \times 3,\ B = 3 \times 3 \quad \text{and } C = 1 \times 4.$$

Basic Matrix Operations

Addition (or subtraction) of matrices is performed by adding (or subtracting) elements in corresponding positions. Addition is only valid if the two matrices have the same order.

Examples:

(i) $\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2+3 & -4+4 & 0+(-1) \\ -1+7 & 3+0 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -1 \\ 6 & 3 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3-1 & 4-7 \\ -2-(-8) & 0-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 6 & -1 \end{bmatrix}$.

(iii) $\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix}$ cannot be done as the orders are different.

When a matrix is multiplied by a real number (called a *scalar*), each element is multiplied by the scalar. The result is another matrix of the same order.

Examples:

(i) $4\begin{bmatrix} 2 & 1 \\ -3 & 9 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4\times 2 & 4\times 1 \\ 4\times -3 & 4\times 9 \\ 4\times 0 & 4\times -5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -12 & 36 \\ 0 & -20 \end{bmatrix}$

(ii) $\frac{1}{2}\begin{bmatrix} 7 & 8 & -10 & 6 & 0.4 \end{bmatrix} = \begin{bmatrix} 3.5 & 4 & -5 & 3 & 0.2 \end{bmatrix}$

(iii) $2\begin{bmatrix} 5 & -3 \\ 0 & -6 \end{bmatrix} - 3\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ 0 & -12 \end{bmatrix} - \begin{bmatrix} 9 & 12 \\ -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & -18 \\ 3 & -33 \end{bmatrix}$

Evaluate the given matrix expression, if possible, using these matrices.

$A = [\;\;1]$ [illegible] $\quad B = [3 \;\; 5] \quad C = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & -4 & 0 \\ 2 & -3 & 5 \end{bmatrix} \quad H = \begin{bmatrix} 5 & 2 & -3 & 0 \\ 1 & 0 & 3 & -1 \\ 4 & -2 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

1. $C + D$ 2. $D - C$ 3. $A + B$ 4. $C + G$ 5. $E + 2F$ 6. $-2E + 5E$
7. BA 8. AB 9. CD 10. DC 11. CA 12. BC
13. GE 14. EG 15. GJ 16. EF 17. HJ 18. $(DA)C$
19. GH 20. $(\frac{1}{2}D)J$ 21. $F\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 22. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}F$ 23. I_3H 24. EI_3

Verify the matrix equation using these matrices.

$A = \begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 6 \\ 5 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$

25. $B + C = C + B$
26. $3(AB) = (3A)B = A(3B)$
27. $A + (B + C) = (A + B) + C$
28. $A(B + C) = AB + AC$
29. $(A + B)C = AC + BC$
30. $(AB)C = A(BC)$

31. Verify that $(AB)C = A(BC)$ using

$A = \begin{bmatrix} 1 & -2 \\ 3 & 3 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 6 & 8 \\ 2 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

32. Arrange the four matrices with the indicated dimensions so that a product of all four can be formed. What will be the dimensions of this product?

$A_{4\times1}, \quad B_{3\times5}, \quad C_{1\times3}, \quad D_{2\times4}$

33. Let $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(a) Evaluate EA.
(b) Compare the answer in part (a) to matrix A and find a matrix F so that FA is the same as A except that the first and third rows are interchanged.

34. (a) Use E and A in Exercise 33 and evaluate AE.
(b) Find a matrix F so that AF is the same as A except that the second and third columns are interchanged.

35. Find matrices E and F so that for matrix A in Exercise 33,

$$EA = \begin{bmatrix} a & b & c \\ d & e & f \\ g + 3a & h + 3b & i + 3c \end{bmatrix} \quad \text{and} \quad AF = \begin{bmatrix} a & b - c & c \\ d & e - f & f \\ g & h - i & i \end{bmatrix}$$

36. Find matrices E and F so that for matrix A in Exercise 33,

$$EA = \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ \end{bmatrix} \quad \text{and} \quad AF = \begin{bmatrix} a & 2b & c \\ d & 2e & f \\ \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) Explain why it is not possible to find x, y so that

$\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

38. Find a, b, c, and d so that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2$ and verify that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = I_2$.

39. (a) Let $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and evaluate $A^2 = AA$.

1

Evaluate the given matrix expression, if possible, using these matrices.

$A = \begin{bmatrix} \text{[illegible]} \\ 1 \end{bmatrix}$ $\quad B = [3 \quad 5] \quad C = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & -4 & 0 \\ 2 & -3 & 5 \end{bmatrix} \quad H = \begin{bmatrix} 5 & 2 & -3 & 0 \\ 1 & 0 & 3 & -1 \\ 4 & -2 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

1. $C + D$	2. $D - C$	3. $A + B$	4. $C + G$	5. $E + 2F$	6. $-2E + 5E$
7. BA	8. AB	9. CD	10. DC	11. CA	12. BC
13. GE	14. EG	15. GJ	16. EF	17. HJ	18. $(DA)C$
19. GH	20. $(\frac{1}{2}D)J$	21. $F\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	22. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}F$	23. I_3H	24. EI_3

Verify the matrix equation using these matrices.

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 6 \\ 5 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$$

25. $B + C = C + B$
26. $3(AB) = (3A)B = A(3B)$
27. $A + (B + C) = (A + B) + C$
28. $A(B + C) = AB + AC$
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31. Verify that $(AB)C = A(BC)$ using

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 3 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 6 & 8 \\ 2 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

32. Arrange the four matrices with the indicated dimensions so that a product of all four can be formed. What will be the dimensions of this product?

$$A_{4\times 1}, \quad B_{3\times 5}, \quad C_{1\times 3}, \quad D_{2\times 4}$$

33. Let $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(a) Evaluate EA.
(b) Compare the answer in part (a) to matrix A and find a matrix F so that FA is the same as A except that the first and third rows are interchanged.

34. (a) Use E and A in Exercise 33 and evaluate AE.
(b) Find a matrix F so that AF is the same as A except that the second and third columns are interchanged.

35. Find matrices E and F so that for matrix A in Exercise 33,

$$EA = \begin{bmatrix} a & b & c \\ d & e & f \\ g + 3a & h + 3b & i + 3c \end{bmatrix} \quad \text{and} \quad AF = \begin{bmatrix} a & b - c & c \\ d & e - f & f \\ g & h - i & i \end{bmatrix}$$

36. Find matrices E and F so that for matrix A in Exercise 33,

$$EA = \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ \end{bmatrix} \quad \text{and} \quad AF = \begin{bmatrix} a & 2b & c \\ d & 2e & f \\ \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Explain why it is not possible to find x, y so that

$$\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

38. Find $a, b, c,$ and d so that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2$ and verify that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = I_2$.

39. (a) Let $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and evaluate $A^2 = AA$.

Worksheet 1

1. SHOULD JUNIORS BE ALLOWED TO GO OFF CAMPUS FOR LUNCH? This is the first year juniors are allowed to go off campus for lunch and the matter comes with mixed emotions. 190 students were surveyed. How many of each grade level want to STOP juniors from going off? How many want to ALLOW juniors to go off?

	juniors	seniors
ALLOW	72%	24%
STOP	20%	53%
DON'T CARE	8%	23%

2. The matrices at the right list teachers' salaries in the WCPSS for the 2004-2005 school year through the 2007-2008 school year. The codes at the top of each matrix classify a teacher according to the number of graduate credits earned. The numbers to the left represent a teacher's current year of service.

   Suppose that in 2004, you accept a teaching position. Your beginning code is A, but by the end of the 2007-2008 school year you have earned enough credits to reach code B. What would your total salary be for the first 4 years of service?
   What if you reached code B status by the end of your 2nd year, What would be your total 4 year salary?

2004 -2005

	A	B	C
1	19,894	19,994	20,094
2	20,671	20,771	20,871
3	21,448	21,548	21,648
4	22,225	22,325	22,425

2005 - 2006

	A	B	C
1	20,525	20,625	20,725
2	21,375	21,475	21,575
3	22,225	22,325	22,425
4	23,075	24,025	23,275

2006 - 2007

	A	B	C
1	22,475	22,575	22,67[illegible]
2	23,325	23,425	23,52[illegible]
3	24,175	24,275	24,375
4	25,025	25,125	25,225

2007-2008

	A	B	C
1	24,525	24,625	24,725
2	25,375	25,475	25,575
3	26,225	26,325	26,425
4	27,075	27,175	27,275

3. GETTING A RAISE The hourly wage of an employee at Jack Astor's depends on the employee's job description and the number of months the employee has worked. The matrix at the right shows the wage rates before the employees earned a 5% wage hike. What does the new matrix look like?

Hourly Wage (in dollars)	**3 mos.**	**6 mos.**	**12 mos.**
Regular Employee	5.00	5.10	5.35
Asst. Manager	5.75	5.85	6.10

4. FOOTBALL ANYONE? The matrix to the right shows the standard widths and lengths (in feet) of four types of football fields. What is the difference in the area of a college football field and an arena field?

Football Sizes	**Width**	**Length**
Arena Football	85	198
College Football	160	360
NFL Football	160	360
Canadian Football	195	450

WS#1

WORKSHEET 2

1. Two competing companies offer cable television to a city with 100,000 households. Company A has 25,000 subscribers and Company B has 30,000. (The other 45,000 households do not subscribe.) The percent changes in cable subscriptions each year are shown below. Explain how matrix multiplication can be used to find the number of subscribers each company will have in one year.

Percent Changes	**From Company A**	**From Company B**	**From Non-subscriber**
To Company A	0.70	0.15	0.15
To Company B	0.20	0.80	0.15
Non-subscriber	0.10	0.05	0.70

2. ROSES, CARNATIONS, AND LILIES You and your cousin decide to make two bouquets: one for your mother and one for your grandparents. The number of roses, carnations, and lilies that will be in your mother's and your grandparents' bouquets is shown in the matrix below. Each rose costs \$3, each carnation costs \$1.25, and each lily costs \$4. Your cousin agrees to share the cost of your grandparents' bouquet. Show how matrix multiplication can be used to determine your cousin's share of the cost.

Number of Flowers	**Roses**	**Carnations**	**Lilies**
Mother's Bouquet	4	5	3
Grandparents' Bouquet	6	4	3

3. EATING HEALTHY The matrix below shows the results of a survey taken by USA Today. In the survey, people were asked why they tried to eat a healthful diet. The survey results show the percent of answers by 8900 women and 7000 men. Show how matrix multiplication can be used to find how many people answered in each of the three categories.

Why Eat Healthy?	**Women**	**Men**
To look better	0.6	0.4
To live longer	0.36	0.53
I don't eat healthy	0.04	0.07

WS #2

EXERCISES 7.3

Find A^{-1}, if it exists, and verify that $AA^{-1} = A^{-1}A = I_n$.

1. $A = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ 0 & \frac{1}{4} \end{bmatrix}$

3. $A = \begin{bmatrix} \frac{1}{3} & -\frac{4}{3} \\ -2 & 8 \end{bmatrix}$

4. $A = \begin{bmatrix} -\frac{3}{2} & \frac{5}{3} \\ \frac{9}{4} & -\frac{5}{2} \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -5 \\ -3 & 4 \end{bmatrix}$

6. $A = \begin{bmatrix} 10 & 15 \\ -5 & -1 \end{bmatrix}$

7. $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

8. $A = \begin{bmatrix} -3 & 6 \\ 6 & -3 \end{bmatrix}$

9. $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, $ab \neq 0$

10. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 4 & 4 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 4 \\ 1 & 7 & 2 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} 4 & -3 & 1 \\ 0 & -1 & 9 \\ -2 & 1 & 4 \end{bmatrix}$

16. $A = \begin{bmatrix} 8 & -13 & 2 \\ -4 & 7 & -1 \\ 3 & -5 & 1 \end{bmatrix}$

17. $A = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \\ 2 & 3 & -1 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & -1 \\ 0 & 2 & 0 & -2 \\ 2 & 0 & 0 & 5 \end{bmatrix}$

20. $A = \begin{bmatrix} 2 & -3 & 0 & 4 \\ -4 & 1 & -1 & 0 \\ -2 & 7 & -2 & 12 \\ 10 & 0 & 3 & -4 \end{bmatrix}$

Write the system of linear equations obtained from the matrix equation $AX = C$ for the given matrices.

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} 9 \\ 14 \end{bmatrix}$

22. $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 4 & 1 \\ 2 & -1 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $C = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix}$

Solve the linear system using the inverse of the matrix of coefficients. Observe that each matrix of coefficients is one of the matrices given in Exercises 1–20.

23. $\begin{aligned} 2x - 5y &= 7 \\ -3x + 4y &= -14 \end{aligned}$

24. $\begin{aligned} 2x - 5y &= -21 \\ -3x + 4y &= -7 \end{aligned}$

25. $\begin{aligned} \tfrac{1}{3}x + \tfrac{2}{3}y &= -8 \\ \tfrac{2}{3}x + \tfrac{1}{3}y &= 5 \end{aligned}$

26. $\begin{aligned} \tfrac{1}{3}x + \tfrac{2}{3}y &= 0 \\ \tfrac{2}{3}x + \tfrac{1}{3}y &= 1 \end{aligned}$

27. $\begin{aligned} x \phantom{{}-y} + 2z &= 4 \\ 2x - y \phantom{{}+2z} &= -8 \\ 3y + 4z &= 0 \end{aligned}$

28. $\begin{aligned} x \phantom{{}-y} + 2z &= -2 \\ 2x - y \phantom{{}+2z} &= 2 \\ 3y + 4z &= 1 \end{aligned}$

29. $\begin{aligned} x + y - z &= 1 \\ x - y - z &= 2 \\ -x - y - z &= 3 \end{aligned}$

30. $\begin{aligned} x + y - z &= -4 \\ x - y - z &= 6 \\ -x - y - z &= 10 \end{aligned}$

31. $\begin{aligned} 8x - 13y + 2z &= 1 \\ -4x + 7y - z &= 3 \\ 3x - 5y + z &= -2 \end{aligned}$

32. $\begin{aligned} -11x + 2y + 2z &= 0 \\ -4x \phantom{{}+2y} + z &= 5 \\ 6x - y - z &= -1 \end{aligned}$

33. $\begin{aligned} w + x \phantom{{}+2y} + 2z &= 2 \\ -w \phantom{{}+x} + 2y - z &= -6 \\ 2x \phantom{{}+2y} - 2z &= 0 \\ 2w \phantom{{}+x+2y} + 5z &= 8 \end{aligned}$

34. $\begin{aligned} w + x \phantom{{}+2y} + 2z &= 1 \\ -w \phantom{{}+x} + 2y - z &= 3 \\ 2x \phantom{{}+2y} - 2z &= 2 \\ 2w \phantom{{}+x+2y} + 5z &= -1 \end{aligned}$

35. For $A = \begin{bmatrix} 2 & 1 \\ \ldots \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ \ldots \end{bmatrix}$ show that $(AB)^{-1} = B^{-1}A^{-1}$

WORKSHEET 3

1. The following matrix equation describes the population changes in a city and its suburbs between 1980 and 1990.

$$\underset{A}{\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} + \underset{B}{\begin{bmatrix} 22{,}000 \\ 37{,}000 \end{bmatrix}} - \underset{C}{\begin{bmatrix} 42{,}500 \\ 27{,}500 \end{bmatrix}} = \underset{D}{\begin{bmatrix} 211{,}200 \\ 282{,}200 \end{bmatrix}} \begin{matrix} \textbf{City} \\ \textbf{Suburbs} \end{matrix}$$

The first row of each matrix refers to the city population, and the second row refers to the suburban population.

A: Transition matrix (shows population changes between city and suburbs)
X: 1980 population matrix
B: New births and people who moved in from another area
C: Deaths and people who moved away to another area
D: 1990 population matrix

How many people lived in the city and its suburbs in 1980? Is the city losing population to the suburbs? Explain.

2. In 1989, Rod Woodson led the National Football League on kick-off returns with 982 yards in 36 carries. Let x represent the total yardages for Woodson during the first half of the 1989 season, and let y represent his total return yardages during the second half of the season. Use an inverse matrix to solve the followin linear system for x and y.

$$\begin{aligned} x + y &= 982 \\ 3x - 2y &= 461 \end{aligned}$$

3. Gold jewelry is seldom made of pure gold because pure gold is soft and expensive. Instead, gold is mixed with other metals to produce a harder, less expensive gold alloy. The amount of gold (by weight) in an alloy is measured in karats. Anything made of 24-karat gold is 100% gold. An 18-karat gold mixture is 75% gold, and so on. Three gold alloys contain the percents of gold, copper, and solver shown in the matrix. You have 20,144 grams of gold, 766 grams of copper, and 1990 grams of silver. How much of each alloy can you make?

Percent by Weight	**Alloy X**	**Alloy Y**	**Alloy Z**
Gold	94%	92%	80%
Copper	4%	2%	4%
Silver	2%	6%	16%

WS#3

EXERCISES 7.4

Evaluate each determinant.

(1.) $\begin{vmatrix} 5 & -1 \\ -3 & 4 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ 3. $\begin{vmatrix} 17 & -3 \\ 20 & 2 \end{vmatrix}$ (4.) $\begin{vmatrix} -7 & 9 \\ -5 & 5 \end{vmatrix}$

5. $\begin{vmatrix} 10 & 5 \\ 6 & -3 \end{vmatrix}$ 6. $\begin{vmatrix} 6 & 11 \\ 0 & -9 \end{vmatrix}$ (7.) $\begin{vmatrix} 16 & 0 \\ -9 & 0 \end{vmatrix}$ 8. $\begin{vmatrix} a & b \\ 3a & 3b \end{vmatrix}$

Solve each system using Cramer's rule.

9. $\begin{aligned} 3x + 9y &= 15 \\ 6x + 12y &= 18 \end{aligned}$

10. $\begin{aligned} x - y &= 7 \\ -2x + 5y &= -8 \end{aligned}$

(11.) $\begin{aligned} -4x + 10y &= 8 \\ 11x - 9y &= 15 \end{aligned}$

(12.) $\begin{aligned} 7x + 4y &= 5 \\ -x + 2y &= -2 \end{aligned}$

13. $\begin{aligned} 5x + 2y &= 3 \\ 2x + 3y &= -1 \end{aligned}$

14. $\begin{aligned} 3x + 3y &= 6 \\ 4x - 2y &= -1 \end{aligned}$

15. $\begin{aligned} \tfrac{1}{3}x + \tfrac{3}{8}y &= 13 \\ x - \tfrac{9}{4}y &= -42 \end{aligned}$

16. $\begin{aligned} x - 3y &= 7 \\ -\tfrac{1}{2}x + \tfrac{1}{4}y &= 1 \end{aligned}$

17. $\begin{aligned} 3x + y &= 20 \\ y &= x \end{aligned}$

18. $\begin{aligned} 3x + 2y &= 5 \\ 2x + 3y &= 0 \end{aligned}$

19. $\begin{aligned} \tfrac{1}{2}x - \tfrac{2}{7}y &= -\tfrac{1}{2} \\ -\tfrac{1}{3}x - \tfrac{1}{2}y &= \tfrac{31}{6} \end{aligned}$

20. $\begin{aligned} -4x + 3y &= -20 \\ 2x + 6y &= -15 \end{aligned}$

21. $\begin{aligned} 9x - 12 &= 4y \\ 3x + 2y &= 3 \end{aligned}$

22. $\begin{aligned} \frac{x - y}{3} - \frac{y}{6} &= \frac{2}{3} \\ 22x + 9(y - 2x) &= 8 \end{aligned}$

Verify that the determinant of the coefficients is zero for each of the following. Then decide whether the system is dependent or inconsistent.

23. $\begin{aligned} 5x - 2y &= 3 \\ -15x + 6y &= -4 \end{aligned}$

24. $\begin{aligned} 2x - 3y &= 5 \\ 10 - 4x &= -6y \end{aligned}$

25. $\begin{aligned} 16x - 4y &= 20 \\ 12x - 3y &= 15 \end{aligned}$

26. $\begin{aligned} 2x - 6y &= -12 \\ -3x + 9y &= 18 \end{aligned}$

27. $\begin{aligned} 3x &= 5y - 10 \\ 6x - 10y &= -25 \end{aligned}$

28. $\begin{aligned} 10y &= 2x - 4 \\ x - 5y &= 2 \end{aligned}$

When variables are used for some of the entries in the symbolism of a determinant, the determinant itself can be used to state equations. Solve for x.

29. $\begin{vmatrix} x & 2 \\ 5 & 3 \end{vmatrix} = 8$ 30. $\begin{vmatrix} 7 & 3 \\ 4 & x \end{vmatrix} = 15$ 31. $\begin{vmatrix} -2 & 4 \\ x & 3 \end{vmatrix} = -1$

Solve each system.

32. $\begin{vmatrix} x & y \\ 3 & 2 \end{vmatrix} = 2$, $\begin{vmatrix} x & -1 \\ y & 3 \end{vmatrix} = 14$

33. $\begin{vmatrix} x & y \\ 2 & 4 \end{vmatrix} = 5$, $\begin{vmatrix} 1 & y \\ -1 & x \end{vmatrix} = -\tfrac{1}{2}$

34. $\begin{vmatrix} 3 & x \\ 2 & y \end{vmatrix} = 13$, $\begin{vmatrix} 3 & 2 \\ y & x \end{vmatrix} = -12$

35. Show that if the rows and columns of a second-order determinant are interchanged, the value of the determinant remains the same.

36. Show that if one of the rows of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a multiple of the other, then the determinant is zero. (*Hint:* Let $a_2 = ka_1$ and $b_2 = kb_1$.)

37. Use Exercises 35 and 36 to demonstrate that the determinant is zero if one column is a multiple of the other.

38. If a common factor k is factored out of each element of a row or column of a two-by-two matrix A, resulting in a matrix B, then $|A| = k|B|$.

39. Make repeated use of the result in Exercise 38 to show the following:

$$\begin{vmatrix} 27 & 3 \\ 105 & -75 \end{vmatrix} = (45)\begin{vmatrix} 9 & 1 \\ 7 & -5 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 27 & 3 \\ 105 & -75 \end{vmatrix} = (45)\begin{vmatrix} 3 & 1 \\ 7 & -15 \end{vmatrix}$$

Then evaluate each side to check.

40. Prove: $\begin{vmatrix} a_1 + t_1 & b_1 \\ a_2 + t_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} t_1 & b_1 \\ t_2 & b_2 \end{vmatrix}$

(3)

... element of a row (or column) of a second-order determinant we add

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EXERCISES 7.5

1. For $A = \begin{bmatrix} 6 & -2 & -1 \\ 0 & -9 & 4 \\ -3 & 5 & 1 \end{bmatrix}$ evaluate $|A|$ using each of these methods.
   (a) Expand by minors along the first column.
   (b) Expand by minors along the third row.
   (c) Rewrite the first two columns.

2. For $A = \begin{bmatrix} -8 & -1 & 0 \\ 4 & 7 & -5 \\ 3 & 0 & 2 \end{bmatrix}$ evaluate $|A|$ using each of these methods.
   (a) Expand by minors along the second column.
   (b) Expand by minors along the second row.
   (c) Rewrite the first two columns.

Evaluate each determinant.

3. $\begin{vmatrix} 2 & 2 & -1 \\ -1 & 3 & -3 \\ 1 & 2 & 3 \end{vmatrix}$
4. $\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -3 & 0 & 4 \end{vmatrix}$
5. $\begin{vmatrix} 1 & -3 & 2 \\ -5 & 2 & 0 \\ 4 & -1 & 3 \end{vmatrix}$
6. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$
7. $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$
8. $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & -5 \\ 3 & 3 & 6 \end{vmatrix}$

Solve for x.

9. $\begin{vmatrix} -1 & x & -1 \\ x & -3 & 0 \\ -3 & 5 & -1 \end{vmatrix} = 0$
10. $\begin{vmatrix} x & 5 & 2x \\ 2x & 0 & x^2 \\ 1 & -1 & 2 \end{vmatrix} = 0$
11. $\begin{vmatrix} 5-x & 0 & -2 \\ 4 & -1-x & 3 \\ 2 & 0 & 1-x \end{vmatrix} = 0$

Evaluate the following for this system:

$$\begin{aligned} 3x - y + 4z &= 2 \\ -5x + 3y - 7z &= 0 \\ 7x - 4y + 4z &= 12 \end{aligned}$$

12. D 13. D_x 14. $\dfrac{D_x}{D}$ 15. $\dfrac{D_y}{D}$

Use Cramer's rule to solve each system.

16. $\begin{aligned} x + y + z &= 2 \\ x - y + 3z &= 12 \\ 2x + 5y + 2z &= -2 \end{aligned}$
17. $\begin{aligned} x + 2y + 3z &= 5 \\ 3x - y &= -3 \\ -4x + z &= 6 \end{aligned}$
18. $\begin{aligned} x - 8y - 2z &= 12 \\ -3x + 3y + z &= -10 \\ 4x + y + 5z &= 2 \end{aligned}$
19. $\begin{aligned} 2x + y &= 5 \\ 3x - 2z &= -7 \\ -3y + 8z &= -5 \end{aligned}$
20. $\begin{aligned} 4x - 2y - z &= 1 \\ 2x + y + 2z &= 9 \\ x - 3y - z &= \tfrac{3}{2} \end{aligned}$
21. $\begin{aligned} 6x + 3y - 4z &= 5 \\ \tfrac{3}{2}x + y - 4z &= 0 \\ 3x - y + 8z &= 5 \end{aligned}$

A general property of determinants is stated in each exercise. Prove this property for the indicated special case.

22. *If a square matrix A contains a row of zeros or a column of zeros, then $|A| = 0$.* Prove this case:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

MATRIX REVIEW

$A = \begin{bmatrix} -1 & 2 \\ 5 & -7 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 5 & -11 \\ 4 & 6 & 0 \end{bmatrix}$ $C = \begin{bmatrix} -2 & 7 \\ -10 & 9 \end{bmatrix}$ $D = \begin{bmatrix} 3 \\ 8 \\ 19 \end{bmatrix}$

1. Find the value of each:

   a. $A + D$ c. ¼ B e. BD g. A^{-1}

   b. $A - C$ d. $2D + B$ f. $(A + C)B$ h. B^{-1}

2. Solve each system using inverse matrices. Matrix to write down your matrix equation.

   a. $(1/3)x + (2/3)y = -8$ (by hand)
   $(2/3)x + (1/3)y = 5$

   b. $w + x + 2z = 2$
   $-w + 2y - z = -6$ (using a calculator)
   $2x - 2z = 0$
   $2w + 5z = 8$

3. Solve each system using Cramer's Rule. Write all matrices on paper.

   a. $-3x + 5y + z = 5$
   $-9y - 4z = 1$
   $6x - 2y - z = 0$

   b. $-7x + 9y = 0$
   $-5x + 5y = 7$

   c. $2x + y = 5$
   $3x - 2z = -7$
   $-3y + 8z = -5$

4. Solve each system using row operations.

   a. $(1/3)x + (2/3)y = -8$
   $(2/3)x + (1/3)y = 5$

   b. $2x + 14y - 4z = -2$
   $-4x - 3y + z = 8$
   $3x - 5y + 6z = 7$

5. Find the additive inverse of $\begin{bmatrix} -3 & -2 & -1 \\ 0 & 4 & 5 \end{bmatrix}$

6. Find the multiplicative inverse of: a) $\begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix}$ b. $\begin{bmatrix} -2 & -3 & 4 \\ 5 & 6 & 2 \end{bmatrix}$