Unit 2 → Matrices

What are they and why do I care?

Matrix → An array of numbers, written in a set of brackets [ ], arranged in rows and columns.

Main use can be found in computer programming.

What does a matrix look like?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2 rows and 3 columns

$A_{2\times3}$

$$B = \begin{bmatrix} 4 & -6 & 2 \\ 8 & 0 & -1 \\ 5 & 10 & 11 \end{bmatrix}$$

Dimensions of a Matrix

row × column

$B_{3\times3}$

Asking for a specific element

What number is in the first row, second column?

$a_{12} = 2$ $b_{23} = -1$

Adding Matrices

<u>MUST</u> have same ~~dimensions~~ dimensions

Add elements that are in corresponding positions

Ex) $T = \begin{bmatrix} 7 & 4 & 9 \\ -6 & 8 & -4 \end{bmatrix}$ $P = \begin{bmatrix} -1 & -4 & 2 \\ 8 & 0 & -6 \end{bmatrix}$

$$[T] + [P] = \begin{bmatrix} 6 & 0 & 11 \\ 2 & 8 & -10 \end{bmatrix}$$

is matrix addition commutative?

$T + P = P + T$

yes!

Subtraction

[same as above!]

NOT COMMUTATIVE!

ex) $T - P = \begin{bmatrix} 7 & 4 & 9 \\ -6 & 8 & -4 \end{bmatrix} - \begin{bmatrix} -1 & -4 & 2 \\ 8 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 7 \\ -14 & 8 & 2 \end{bmatrix}$

$$P - T = \begin{bmatrix} -8 & -8 & -7 \\ 14 & -8 & -2 \end{bmatrix}$$

Matrix Multiplication

(not exactly easy)

DIMENSIONS ARE IMPORTANT

$\underbrace{2 \times \overbrace{3 \cdot 3}^{\text{must match!}} \times 5}_{\text{dimensions of answer.}}$

$\underbrace{4 \times 1 \cdot 2}_{\text{Ahhhhh!}} \times 4$

Can't be multiplied

$$\begin{bmatrix} 4 & -1 & 3 \\ 0 & 8 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 & 1 & 9 \\ 1 & -1 & -3 & 3 & 10 \\ 2 & 0 & 7 & -5 & 1 \end{bmatrix} = \begin{bmatrix} \underline{9} & \underline{-7} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{4} & \underline{\quad} & \underline{\quad} & \underline{34} & \underline{\quad} \end{bmatrix}$$

Row 1 · Column 1: $4(1) + (-1)(1) + (3)(2) = 9$

Row 1 · column 2 $= (4)(-2) + (-1)(-1) + (3)(0) = -7$

Row 2 · column 1 $= (0)(1) + (8)(1) + (-2)(2) = 4$

Row 2 · column 4 $= (0)(1) + (8)(3) + (-2)(-5) = 34$

Matrix Properties

ADD

Identity: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

↖ inverse matrix

MULT.

identity: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ... etc

Must be a square Matrix!

inverse → find using calculator!

Not all square matrices have inverses.

$\frac{\cancel{4}x}{\cancel{4}} = \frac{16}{4}$

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Inverses in Action!

$2 \cdot \frac{1}{2} = 1$

$\frac{2}{3} \cdot \frac{3}{2} = 1$

$$AA^{-1} = I$$

matrix · inverse = identity.

Ex) $\begin{bmatrix} 4 & 9 & -2 \\ 7 & 4 & 1 \\ -3 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ 28 \end{bmatrix}$

$$\cancel{\begin{bmatrix} \ \ \end{bmatrix}^{-1} \begin{bmatrix} 4 & 9 & 2 \\ 7 & 4 & 1 \\ -3 & 2 & 9 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \ \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 18 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \ \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 18 \\ 28 \end{bmatrix}$$

$x = 1 \quad y = 2 \quad z = 3$

ex) $\begin{bmatrix} -2 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \end{bmatrix}$

$$A \cdot x = B$$

$$x = A^{-1}B$$

ex) $\begin{bmatrix} 2 & 9 & -3 \\ -3 & -1 & 0 \\ 1 & 10 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \\ -9 \end{bmatrix}$

Finding the determinant of a Square matrix

only square matrices have determinants

What is it? A number that has multiple uses (calculate area, volume, ..)

How do we find it? 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ad - cb$$

Notation: $|A| = ad - cb$

ex:

$$B = \begin{bmatrix} 4 & -2 \\ 8 & 3 \end{bmatrix}$$

★ $|B| = (4)(3) - (8)(-2)$

$12 + {}^{+}16 = \boxed{28}$

if the determinant is zero, then matrix does not have An inverse ☹

$E = \begin{bmatrix} 7 & 2 \\ 1 & 0 \end{bmatrix}$ $\quad F = \begin{bmatrix} -9 & -1 \\ 2 & -3 \end{bmatrix}$ $\quad G = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$

Find $|E| = 7(0) - (1)(2) = -2$

$|F| = (-9)(-3) - (2)(-1) = 29$

$|G| = (2)(-6) - (4)(-3) = 0$

Which matrices have inverses? [E] and [F]

3×3 Determinants

Method 1: Diagonals

ex)
$$\left|\begin{array}{ccc|cc} 1 & -2 & 3 & 1 & -2 \\ -1 & 0 & 4 & -1 & 0 \\ 6 & 1 & -3 & 6 & 1 \\ & & & \uparrow & \uparrow \\ & & & 1^{st} & 2^{nd} \end{array}\right.$$

Column

$(1 \cdot 0 \cdot -3) + (-2 \cdot 4 \cdot 6) + (3 \cdot -1 \cdot 1) = -51$

$(6 \cdot 0 \cdot 3) + (1 \cdot 4 \cdot 1) + (-3 \cdot -1 \cdot -2) = -2$

$-51 - (-2) = -49$

Cramer's Rule

used to solve a system of equations.

$$2x + 4y = \boxed{10}$$
$$x - 3y = \boxed{-5}$$

$$D = \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = -6 - 4 = -10$$

$$D_x = \begin{vmatrix} \boxed{10} & 4 \\ \boxed{-5} & -3 \end{vmatrix} = -30 + 20 = \checkmark -10$$

$$D_y = \begin{vmatrix} 2 & \boxed{10} \\ 1 & \boxed{-5} \end{vmatrix} = -10 - 10 = -20$$

$$x = \frac{D_x}{D} = \frac{-10}{-10} \quad y = \frac{D_y}{D} = \frac{-20}{-10}$$

$$\boxed{x = 1 \quad y = 2}$$

Ex 2)

$$-4x + 3y = -33$$
$$2x - 4y = 34$$

Solve using Cramer's Rule.

$$D = \begin{vmatrix} -4 & 3 \\ 2 & -4 \end{vmatrix} \quad D_x = \begin{vmatrix} -33 & 3 \\ 34 & -4 \end{vmatrix} \quad D_y = \begin{vmatrix} -4 & -33 \\ 2 & 34 \end{vmatrix}$$

$$D = 10 \qquad D_x = 30 \qquad D_y = -70$$

$$x = \frac{30}{10} \qquad y = \frac{-70}{10}$$

$$\boxed{x = 3 \quad y = -7}$$

32. $2x-3y=2$
$3x+y=14$

$D=\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}=11$ $D_x=\begin{vmatrix} 2 & -3 \\ 14 & 1 \end{vmatrix}=44$

$D_y=\begin{vmatrix} 2 & 2 \\ 3 & 14 \end{vmatrix}=22$

$\boxed{x=4 \quad y=2}$

33. $4x+2y=5$
$x+y=-\frac{1}{2}$

$D=\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}=2$ $D_x=\begin{vmatrix} 5 & 2 \\ -\frac{1}{2} & 1 \end{vmatrix}=6$

$D_y=\begin{vmatrix} 4 & 5 \\ 1 & -1/2 \end{vmatrix}=$ $-2-5=-7$

34. ~~$3y-2x=13$~~
$3x-2y=-12$
$-2x+3y=13$

$D=\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}=13$ $D_x=\begin{vmatrix} -12 & -2 \\ 13 & 3 \end{vmatrix}=-10$

$D_y=\begin{vmatrix} 3 & -12 \\ -2 & 13 \end{vmatrix}=15$